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Forced vertical vibrations of an elastic elliptic plate on an elastic half space – a direct approach using orthogonal polynomials

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Abstract

The forced vibration of an elastic half space produced by a rigid elliptic indenter oscillating about an axis perpendicular to the plane face of the half space is considered. The boundary conditions lead to a two-dimensional dual integral equation in terms of the unknown normal stress. By appropriate substitution, the dual integral equation is first reduced to a two-dimensional Fredholm integral equation. This is transformed to an infinite set of equations using Abelian transformations. Next, the Abel-transformed variable of the unknown normal stress is expanded in terms of orthogonal Jacobi polynomials, and by solving the system of linear equations, orthogonal polynomial solutions are obtained. The method used to obtain the orthogonal polynomial solutions of this problem is new and the major advantage of this expansion technique is that it is valid for all frequencies. Detailed numerical work has been given for the total load on the disc for different values of frequencies.

Keywords: Orthogonal polynomial solution; Flat elliptic indenter; Multilayered elastic half space; Dual integral; Jacobi's polynomials; Transfer matrix

1. Introduction

Problems involving a vibrating elastic body in contact with an elastic medium are of considerable practical interest. To improve dynamic models of buildings and large structures, to ascertain their stability against earthquake vibrations or to design ultrasonic hardness testing equipment, it is essential to investigate the dynamic response of elastic foundations or elastic plates on elastic media. Although a large number of papers have been devoted to the subject, most of them are concerned with the problem where the contact region is circular.

The physical requirement of a uniform displacement under the rigid body and a zero stress at the surface away from the body, leads to a mixed boundary value problem which can be expressed in terms of a two-dimensional dual integral equation.

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In the case of a circular elastic plate, Robertson (1965) and Gladwell (1968) reduced the two-dimensional dual integral equation to a one-dimensional Fredholm integral equation and obtained a series solution. Awojobi and Grootenhuis (1965) solved the dual integral equation by a series of expansion procedure. A direct method is proposed by Krenk and Schmidt (1981) in which the integral equation is solved expressing the vertical stress in terms of Legendre polynomials. The solution implies a direct matrix relation between stresses and deformations. Apart from the numerical results for the low-frequency cases, Luco and Westmann (1971) developed numerical solutions by reducing the Fredholm integral equation to a system of algebraic equations and obtained various dynamic compliances for large values of frequencies.

In order to model the dynamics of various large structures, the results for foundations other than circular ones are required. Foundations generally have elliptic cross-section and the implications for such a geometry are considered in the present paper. Stallybrass and Scherer (1975) used a variational procedure and obtained the analytical expression for the reciprocal of the total load under the elliptic disc. Roy (1986) reconsidered the elliptic geometry problem and reduced the dual integral equation into a Fredholm integral equation of the first kind which could be rearranged in a suitable form after separating out the terms corresponding to the static solutions. Successive low-frequency terms are obtained by perturbing the static solutions. So, for the case of elliptic punches, specific results have only been obtained for low values of dimensionless frequency.

A knowledge of dynamic compliances for a larger frequency range is necessary if structural analysis is to be performed incorporating the effect of soil–structure interaction. In this paper, we reformulate the problem of dynamic compliances for the elliptic punch and obtain an orthogonal polynomial solution from which we compute the numerical results for a significantly wider range of frequencies than previously available.

Following Roy (1986), the two-dimensional dual integral equation involving the unknown stress under the punch is reduced to a Fredholm integral equation of the first kind. Applying suitable transformations, this equation can be converted into an infinite system of equations involving the Abelian transformation of the Fourier coefficients of the unknown stress. Here, we express the unknown variable in terms of orthogonal Jacobi polynomials. Using certain properties of orthogonal polynomials, an infinite set of linear equations involving the coefficients of Jacobi polynomials is formed and by solving this set of equations, the orthogonal polynomial solutions for dynamic compliances of the elliptic disc with vertical vibrations are obtained. Retaining up to second-order terms in frequency, the analytical expressions of the low-frequency expansion of the non-dimensional part of the total load are found to be in complete agreement with the previous work by Stallybrass and Scherer (1975) and by Roy (1986). The advantage of the new method presented here is that we illustrate this for various frequencies, and different aspect ratios of the elliptic geometry. Numerical values of the total load under the disc can be predicted for a wide range of frequencies not previously considered.

2. Basic equations

A rigid elastic elliptic plate with semi-axes a and b is vibrating at an angular frequency ω while remaining in frictionless contact with an elastic half space (Fig. 1). The equation of motion in terms of the displacement vector $\vec{u} = (u_x, u_y, u_z)e^{i\omega t}$ is given by

$$(\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\vec{u} - \mu\,\mathrm{curl}\,\mathrm{curl}\,\vec{u} + \omega^2 \rho_1 \vec{u} = 0, \tag{2.1}$$

where λ and μ are the Lame constants and ρ_1 is the density.

The boundary conditions at $z = 0$ are

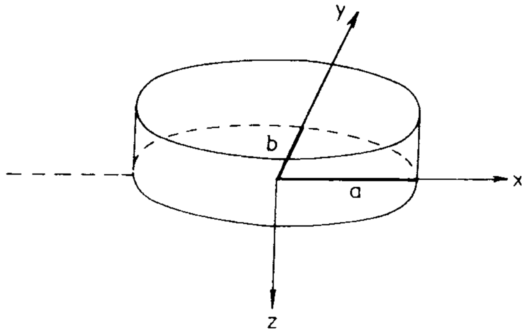

Fig. 1. Elliptic plate on an elastic half space.

$$\begin{aligned} &\tau_{zz} = 0 && \text{for } (x,y) \notin S, \\ &\tau_{zx} = 0 = \tau_{zy} && \forall x,y, \\ &u_z = w_0, && (x,y) \in S, \end{aligned} \tag{2.2}$$

where S is the elliptic contact area and w_0 is the constant amplitude of the vertical vibration produced by the disc.

The solution is given by

$$\vec{u} = \vec{\nabla}\Phi + \vec{\nabla} \times \vec{\nabla} \times \left(\vec{e}_z\Psi\right) + \vec{\nabla} \times \left(\vec{e}_z X\right), \tag{2.3}$$

where Φ, Ψ, X are the potentials, and $\vec{e}_z$ is the unit vector along the normal to the plate.

Using the cartesian co-ordinate system (x, y, z) with the origin at the centre of the disc and z-axis along the normal to the plane, the potentials are of the form

$$\begin{aligned} \Phi &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi,\eta)\, \exp\left[-\,\mathrm{i}(\xi x + \eta y) - v_1 z\right] \mathrm{d}\xi\, \mathrm{d}\eta, \\ \Psi &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(\xi,\eta)\, \exp\left[-\,\mathrm{i}(\xi x + \eta y) - v_2 z\right] \mathrm{d}\xi\, \mathrm{d}\eta, \\ X &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(\xi,\eta)\, \exp\left[-\,\mathrm{i}(\xi x + \eta y) - v_2 z\right] \mathrm{d}\xi\, \mathrm{d}\eta, \end{aligned} \tag{2.4}$$

where

$$v_n = \begin{bmatrix} \left(\lambda_1^2 - k_n^2\right)^{1/2}, & \lambda_1 > k_n, \\ \mathrm{i}\left(k_n^2 - \lambda_1^2\right)^{1/2}, & -k_n < \lambda_1 < k_n, \quad n = 1,2 \\ -\left(\lambda_1^2 - k_n^2\right)^{1/2}, & \lambda_1 \leqslant k_n \end{bmatrix} \tag{2.5}$$

with

$$k_1 = \frac{\rho_1\omega}{\lambda + 2\mu}, \qquad k_2 = \frac{\rho_1\omega}{\mu}, \qquad \lambda_1^2 = \xi^2 + \eta^2. \tag{2.6}$$

The harmonic time dependence $e^{\mathrm{i}\omega t}$ is suppressed from now on. The displacement components u_j and stress components τ_{zj} $(j = x, y, z)$ are obtained from Eq. (2.3) using Eq. (2.4).

3. Derivation of the integral equation

Expressing the boundary conditions in terms of the potentials, we obtain the following two-dimensional dual integral equation (Roy, 1986)

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1(\xi,\eta)\,\exp[-\mathrm{i}(\xi x+\eta y)]\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \quad (x,y)\notin S,\\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{A_1(\xi,\eta)k_2^2 v_1}{2F(\xi,\eta)}\exp[-\mathrm{i}(\xi x+\eta y)]\,\mathrm{d}\xi\,\mathrm{d}\eta = w_0, \quad (x,y)\in S,
\end{aligned}
\tag{3.1}
$$

where

$$
F(\xi,\eta) = \left(\xi^2+\eta^2-k_2^2/2\right)^2 - \left(\xi^2+\eta^2\right)v_1 v_2, \qquad A_1(\xi,\eta) = \frac{F(\xi,\eta)}{v_1}B(\xi,\eta). \tag{3.2}
$$

In order to obtain the solution of Eq. (3.1), we consider the normal stress under the plate is $g(x,y)$, i.e.

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1(\xi,\eta)\,\exp[-\mathrm{i}(\xi x+\eta y)]\,\mathrm{d}\xi\,\mathrm{d}\eta = g(x,y),
$$

where

$$
g(x,y) = 0, \quad (x,y)\notin S. \tag{3.3}
$$

Then, the first equation in Eq. (3.1) is satisfied. Substituting the value of $A_1(\xi,\eta)$ from Eq. (3.3) into the second equation of Eq. (3.1), we get the following Fredholm integral equation of the first kind,

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int\int_{s} \frac{k_2^2 v_1}{2F(\xi,\eta)} g(x',y')\,\exp\left[-\mathrm{i}\{\xi(x-x')+\eta(y-y')\}\right]\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}\xi\,\mathrm{d}\eta = w_0. \tag{3.4}
$$

To represent the above equation in cylindrical polar coordinates, the following transformations are used:

$$
\begin{aligned}
(a\xi, b\eta) &= (k\cos\chi, k\sin\chi), \qquad k\in[0,\infty), \quad \chi\in[0,2\pi],\\
(x,y) &= (ar\cos\theta, br\sin\theta), \qquad r\in[0,1], \quad \theta\in[0,2\pi],\\
(x',y') &= (a\rho\cos\phi, b\rho\sin\phi), \qquad \rho\in[0,1], \quad \phi\in[0,2\pi].
\end{aligned}
\tag{3.5}
$$

Next, the normal stress under the disc $g(\rho,\phi)$ is expressed in terms of Fourier cosine and sine series as

$$
g(\rho,\phi) = \sum_{n=0}^{\infty} g_n^c(\rho)\cos n\phi + \sum_{n=1}^{\infty} g_n^s(\rho)\sin n\phi \tag{3.6}
$$

and some standard representation (Appendix A) involving Bessel functions is used in Eq. (3.4).

Finally, we relate $g_n^{c(s)}(\rho)$ to the new functions $\phi_n^{c(s)}(t)$ through the Abel transformation as follows:

$$
\begin{aligned}
&g_n^{c(s)}(\rho) = A_{n-1}^{-1}\left[\phi_n^{c(s)}(t)\right]\\
&\text{s.t.}\quad \phi_n^{c(s)}(t) = A_{n-1}\left[g_n^{c(s)}(\rho)\right],
\end{aligned}
\tag{3.7}
$$

where

$$
A_{n-1}[f(\rho)] = t^n\int_t^1 \frac{\rho^{1-n}f(\rho)}{\left(\rho^2-t^2\right)^{1/2}}\mathrm{d}\rho, \quad A_{n-1}^{-1}[g(t)] = -\frac{2}{\pi}\rho^n\frac{\partial}{\partial\rho}\int_\rho^1 \frac{t^{1-n}g(t)}{\left(t^2-\rho^2\right)^{1/2}}\mathrm{d}t. \tag{3.8}
$$

Proceeding as in Chatterjee and Roy (1990), we substitute Eqs. (3.5) and (3.6) into Eq. (3.4) and by using Eq. (3.7), the following infinite system of equations is obtained: for $s = 0,1,2,\ldots,\infty$,

$$\frac{1}{4}\sum_{\substack{n=0\\(n+s)-\text{even}}}^{\infty} \mathrm{i}^{s}(-\mathrm{i})^{n}\varepsilon_n\varepsilon_s\int_0^{\infty}\int_0^{2\pi}\int_0^{1} k^2\zeta^{s+1/2}t^{1/2}F_1(k,\chi)\phi_n^c(t)J_{n-1/2}(kt)J_{s-1/2}(k\zeta)\cos n\chi\cos s\chi\,\mathrm{d}k\,\mathrm{d}t\,\mathrm{d}\chi = f_s(\zeta), \tag{3.9}$$

where

$$F_1(k,\chi) = \frac{k_2^2\sqrt{(v^2-k_1^2)}}{(v^2-k_2^2/2)^2 - v^2\sqrt{(v^2-k_1^2)(v^2-k_2^2)}}, \tag{3.10}$$

$$v = \frac{k}{b}\sqrt{(1-k_0^2\cos^2\chi)},$$

$$k_0^2 = \left(1-b^2/a^2\right), \tag{3.11}$$

$$f_s(\zeta) = \pi w_0\frac{\mathrm{d}}{\mathrm{d}\zeta}\int_0^{\zeta}\frac{r^{s+1}}{\sqrt{\left(\zeta^2-r^2\right)}}\mathrm{d}r. \tag{3.12}$$

4. Polynomial representation of the solution

Following Mukherjee (1998), the unknown variable $\phi_n^{c(s)}(t)$ is now expanded in terms of orthogonal Jacobi polynomials as

$$\begin{aligned}\phi_n^c(t) &= \sum_{j=0}^{\infty} W_j^n t^n P_j^{(n-1/2,0)}\left(1-2t^2\right),\\ \phi_n^s(t) &= \sum_{j=0}^{\infty} V_j^n t^n P_j^{(n-1/2,0)}\left(1-2t^2\right),\end{aligned} \tag{4.1}$$

where W_j^n and V_j^n are the Jacobi coefficients.

Substituting Eq. (4.1) into Eq. (3.9) and using some integral expression involving Bessel functions (see Appendix A), we obtain

$$\begin{aligned}\frac{1}{4}\sum_{\substack{n=0\\(n+s)-\text{even}}}^{\infty}\sum_{j=0}^{\infty} \mathrm{i}^{s}(-\mathrm{i})^{n}\varepsilon_n\varepsilon_s W_j^n\int_0^{\infty}\int_0^{2\pi} F_1(k,\chi)J_{n+2j+1/2}(k)J_{s+2m+1/2}(k)\cos s\chi\cos n\chi\,\mathrm{d}k\,\mathrm{d}\chi\\ = \pi\int_0^1 f_s(\zeta)P_m^{(s-1/2,0)}(1-2\zeta^2)\,\mathrm{d}\zeta, \quad \forall s = 0,1,2,\ldots,\infty.\end{aligned} \tag{4.2}$$

Now, using the transformation,

$$(k\cos\chi, k\sin\chi) = (au\cos\psi, bu\sin\psi) \tag{4.3}$$

and applying certain properties of orthogonal polynomials (see Appendix B), Eq. (4.2) becomes, for $s = 0,1,2,\ldots,\infty$,

$$\sum_{\substack{n=0\\(n+s)-\text{even}}}^{\infty}\sum_{j=0}^{\infty} A_{n,j}^{ms} W_j^n = \frac{\pi}{b} w_0 \quad \text{for } s=0 \quad \text{and} \quad m=0,$$
$$= 0 \qquad \text{for } s=0 \quad \text{and} \quad m \neq 0,$$
$$= 0 \qquad \text{for } s \neq 0, \tag{4.4}$$

where

$$A_{n,j}^{ms} = \int_0^\infty \int_0^{\pi/2} \frac{1}{p} F_2(u) J_{n+2j+1/2}(aup) J_{s+2m+1/2}(aup) \times \cos\left\{n \tan^{-1}\left(\frac{b}{a}\tan\psi\right)\right\} \cos\left\{s \tan^{-1}\left(\frac{b}{a}\tan\psi\right)\right\} \mathrm{d}u\,\mathrm{d}\psi \tag{4.5}$$

and

$$p = \sqrt{(1 - k_0^2 \sin^2\psi)}, \tag{4.6}$$

$$F_2(u) = \frac{k_2^2 (u^2 - k_1^2)^{1/2}}{(u^2 - k_2^2/2)^{1/2} - u^2 (u^2 - k_1^2)^{1/2} (u^2 - k_2^2)^{1/2}}. \tag{4.7}$$

Following Krenk and Schmidt (1982) by contour integration in the complex plane, $A_{n,j}^{ms}$ is reduced to the following form:

$$A_{n,j}^{ms} = \mathrm{i}\int_0^{\pi/2} \frac{1}{p} \left[\int_0^\gamma \frac{(t^2-\gamma^2)^{1/2}}{(t^2-1/2)^2 + t^2(\gamma^2-t^2)^{1/2}(1-t^2)^{1/2}} H_{s+2m+1/2}^{(2)}(apk_2t) J_{n+2j+1/2}(apk_2t)\,\mathrm{d}t \right.$$
$$+ \int_\gamma^1 \frac{(1-t^2)^{1/2} t^2 (t^2-\gamma^2)}{(t^2-1/2)^4 + t^4(t^2-\gamma^2)(1-t^2)} H_{s+2m+1/2}^{(2)}(apk_2t) J_{n+2j+1/2}(apk_2t)\,\mathrm{d}t$$
$$\left. - \frac{\pi(s^2-\gamma^2)^{1/2}}{G'(s)} H_{s+2m+1/2}^{(2)}(apk_2s) J_{n+2j+1/2}(apk_2s) \right] \cos\left\{n \tan^{-1}\left(\frac{b}{a}\tan\psi\right)\right\}$$
$$\times \cos\left\{s \tan^{-1}\left(\frac{b}{a}\tan\psi\right)\right\} \mathrm{d}\psi \quad \text{when } (n+2j) \geqslant (s+2m). \tag{4.8}$$

Here, $\gamma = k_1/k_2$ and s is the real root of the equation:

$$G(t) = 0, \qquad \text{where } G(t) = (t^2 - k_2^2/2)^2 - t^2 (t^2 - k_1^2)^{1/2} (t^2 - k_2^2)^{1/2} \tag{4.9}$$

and $H_n{}^2(\ )$ is the Hankle function of second kind.

For $(n+2j) < (s+2m)$, we have

$$A_{n,j}^{ms} = A_{s,m}^{jn}. \tag{4.10}$$

Thus, solving the equations in (4.4) the coefficients W_j^n of Jacobi polynomials are obtained.

5. Analytical solutions for low frequencies

For the case of low frequencies, we limit the solutions to $o(k_2^2)$. So, the terms of the orthogonal Jacobi polynomials are retained up to order two and we obtain the coefficients W_0^0 and W_1^0 of the Jacobi polynomials solving the first two equations in Eq. (4.4) (see Appendix C). Substituting $s = 0$ and $n = 0$ and expanding the product of Bessel functions in the integrals of $A_{n,j}^{ms}$ in Eq. (4.8), we obtain integrals of the following form:

$$I_n = \int_0^1 \mathscr{K}(t) t^{n-1} \mathrm{d}t - \frac{\pi\sqrt{(s^2-\gamma^2)}}{G'(s)} s^n \quad n = 0, 1, 2, \ldots \tag{5.1}$$

where

$$\mathscr{K}(t) = \begin{bmatrix} \dfrac{t\sqrt{\gamma^2-t^2}}{(t^2-1/2)^2+t^2\sqrt{(\gamma^2-t^2)(1-t^2)}}, & 0 \leqslant t \leqslant \gamma, \\ \dfrac{t^3(t^2-\gamma^2)(1-t^2)^{1/2}}{(t^2-1/2)^4+t^4\sqrt{(t^2-\gamma^2)(1-t^2)}}, & \gamma \leqslant t \leqslant 1. \end{bmatrix} \tag{5.2}$$

The total load to be applied on the disc in order to keep it stable against the given vibration is

$$P = \int\int_s \tau_{zz}(x, y, 0)\,\mathrm{d}x\,\mathrm{d}y = 4ab\mu \int_0^1 \phi_0(t)\,\mathrm{d}t = 4ab\mu W_0^0. \tag{5.3}$$

Substituting the expression of W_0^0 in terms of I_ns, we get

$$P = \frac{2aw_0\pi\mu}{K}\left[1 + \frac{\mathrm{i}ak_2I_1}{2K} + \frac{k_2^2a^2}{2\pi K}\left(\frac{4}{3}I_2E - \frac{I_1^2\pi}{2K}\right) + o\left(k_2^3\right)\right], \tag{5.4}$$

where E and K are the elliptic integrals of first and second kind and $I_0 = \pi$.

Expression (5.4) shows complete agreement with the expression obtained by Roy (1986) and Stallybrass and Scherer (1975) for low frequencies.

6. Numerical results for low and high frequencies

Returning to our direct method, let us consider that f_1 and f_2 correspond to the real and imaginary part of the reciprocal of the non-dimensional load $4\mu aw_0/(1-\nu')P$, i.e.

$$f_1 - \mathrm{i}f_2 = \frac{4\mu aw_0}{(1-\nu')P}. \tag{6.1}$$

Using Eq. (5.3), it can be expressed as

$$f_1 - \mathrm{i}f_2 = \frac{1}{\pi(1-\nu')}\frac{1}{W_0^0} \tag{6.2}$$

where ν' is Poisson's ratio related by

$$\gamma^2 = \frac{k_1^2}{k_2^2} = \frac{1-2\nu'}{2(1-\nu')}.$$

We will now illustrate the numerical calculation of f_1 and f_2 by some examples.

Table 1
Values of f_1 and f_2 retaining six and eight terms of the series expansion of $\phi_n^c(t)$ for $\gamma^2 = 1/4$

$q = b/a$	k_2	Values of f_1		Values of f_2	
		For six terms	For eight terms	For six terms	For eight terms
1/2	4	0.1150948	0.1150947	−0.4780423	−0.4780422
1/2	5	0.06699269	0.0669926	−0.3800982	−0.3800979
1/3	1	1.304853	1.304853	−0.6571907	−0.6571906
1/3	8	0.05460862	0.0546085	−0.3550709	−0.3550707
1/4	2	0.9965307	0.9965306	−0.8717572	−0.8717567
1/4	6	0.2025709	0.2025707	−0.6169359	−0.6169356

6.1. Convergence of our method

We first compute $A_{n,j}^{ms}$ in Eq. (4.8) by direct integration with $\gamma^2 = (1/4)$ and aspect ratio $q = (b/a) = 1, 1/2, 1/3, 1/4$ for different values of dimensionless frequencies k_2 in the range 0–10. Substituting these values in Eq. (4.4), we solve the linear equations to obtain the coefficients W_j^n of Jacobi polynomials.

To test the convergence, we compared the numerical computation retaining six and eight terms of the series expansion of $\phi_n^c(t)$. Table 1 shows that the successive values of f_1 and f_2 for the two cases are nearly equal which implies the convergence of the solutions.

For the case of order six, we compute $A_{n,j}^{ms}$ for $\{s = 0, 2, 4, 6, m = 0, 1, 2, 3, n = 0, 2, 4, 6 \text{ and } j = 0, 1, 2, 3\}$, and 10 algebraic equations from Eq. (4.4) are solved.

6.2. Comparison for low-frequency range

For elliptic plates, applying the principle of variational approximation, Stallybrass and Scherer (1975) computed the values of f_1 and f_2 for low frequencies when $k_2 \leqslant 1$. In Table 2, we compare the values of f_1 and f_2 obtained by their method with the values obtained by our direct method described here for

Table 2
Comparison between the values of f_1 and f_2 obtained by our direct method and approximate method by Stallybrass and Scherer, when $\gamma^2 = 1/4$

	$q = b/a$	1		1/2		1/3		1/4	
	$\kappa = k_2 b$	Direct method	Approximate method	Direct method	Approximate method	Direct method	Approximate method	Direct method	Approximate method
f_1	0.0	0.9990228	1.0000	1.370574	1.3729	1.606897	1.6098	1.779925	1.7833
	0.2	0.9797338	0.9807	1.311844	1.3138	1.488304	1.4910	1.584328	1.5874
	0.5	0.8831638	0.8844	1.044378	1.0453	1.036755	1.0266	0.9965307	–
	0.8	0.7256346	0.7263	0.7083953	–	0.6460894	–	0.6187256	–
	1.0	0.6049319	0.6021	0.5207506	–	0.4695930	–	0.4497818	–
	2.0	0.1683695	–	0.1150948	–	0.1050762	–	0.1016092	–
f_2	0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.2	0.154477	0.1546	0.3031298	0.3036	0.4407426	0.4418	0.562666	0.5650
	0.5	0.361446	0.3624	0.6414468	0.6472	0.802777	0.8141	0.9717572	–
	0.8	0.5090966	0.5137	0.7640374	–	0.8275032	–	0.8432221	–
	1.0	0.564684	0.5737	0.7538653	–	0.7800583	–	0.788866	–
	2.0	0.478448	–	0.4780423	–	0.4757669	–	0.4759291	–

$a = 1, 2, 3, 4$, $b = 1$ and for different values of $\kappa = bk_2$. We notice that the agreement between both the results is quite good for the case of low frequencies.

6.3. Application to high-frequency values

Finally, we illustrate the big advantage of our method. We compute the values of f_1 and f_2 for the case of an elliptic disc over a large range in non-dimensional frequencies $k_2 (0 \leqslant k_2 \leqslant 10)$. For a circular plate, the results completely agree with those obtained by Luco and Westmann (1971) for higher frequencies. However, we are currently not in a position to compare our results with any other method dealing with elliptical geometry.

In Figs. 2 and 3, the values of f_1 and f_2 are plotted against $k_2 (0 \leqslant k_2 \leqslant 10)$ taking $(b/a) = 1, 1/2, 1/3, 1/4$ and $\gamma^2 = (1/4)$. Fig. 2 illustrates that the values of f_1 decrease as k_2 increases and implies that for

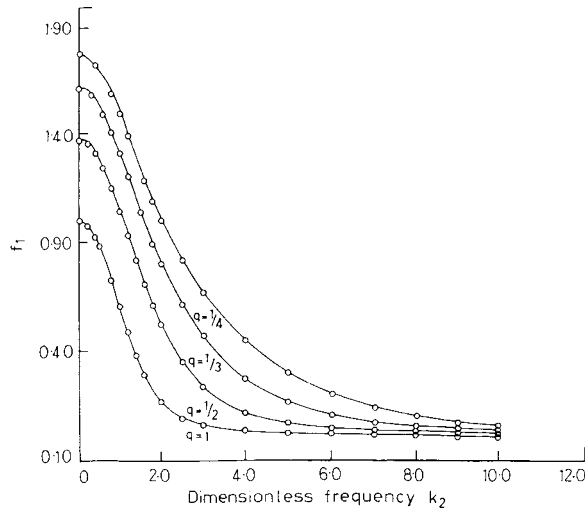

Fig. 2. Plot of f_1 for various ellipses against the values of dimensionless frequency k_2 with $\gamma^2 = 1/4$, where $q = b/a$.

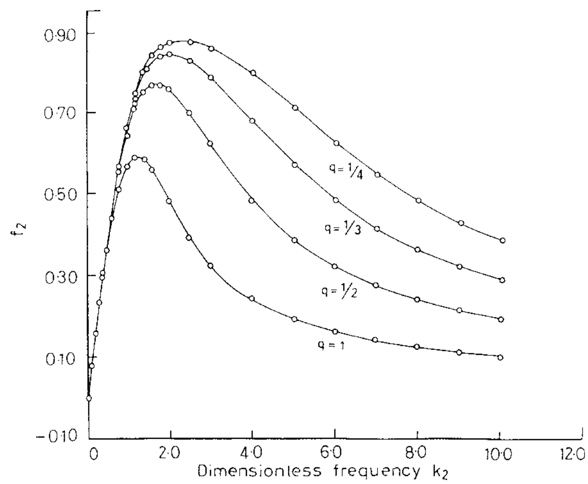

Fig. 3. Plot of f_2 for different ellipses against the values of dimensionless frequency k_2 with $\gamma^2 = 1/4$, where $q = b/a$.

comparatively larger values of frequencies those values of f_1 become constant. In Fig. 3, we notice that the values of f_2 are independent of the aspect ratio q for low frequencies. However at higher frequencies, the aspect ratio appears to have a significant influence on f_2. The peaks of the curves show that the resonance frequency of an elliptic plate is larger than that of a circular plate and the resonance frequency increases as the aspect ratios decrease.

The results obtained also predict that an elliptic disc can bear a larger load than a circular one.

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Appendix A

Some properties of Bessel functions are as follows:

(1) Following standard relation (Gradshteyn and Ryzhik, 1980) is used in Eq. (3.4),

$$\exp[\pm \mathrm{i}z\cos\theta] = \sum_{n=0}^{\infty}\varepsilon_n(\pm\mathrm{i})^n J_n(z)\cos n\theta,$$

$$\text{where } \varepsilon_n = \begin{bmatrix} 1, & n=0, \\ 2, & n>0. \end{bmatrix}$$

2. The integral expression mentioned in Section 4 is

$$\int_0^1 x^{s+1/2} P_m^{(s-1/2,0)}(1-2x^2) J_{s-1/2}(xy)\,\mathrm{d}x = \frac{J_{s+2m+1/2}(y)}{y}.$$

Appendix B

On the right-hand side of Eq. (4.2) using Eq. (3.12), the following expression is obtained:

For $s = 0$,

$$\int_0^1 \left\{ \frac{\mathrm{d}}{\mathrm{d}\zeta} \int_0^{\zeta} \frac{r}{\sqrt{\zeta^2 - r^2}}\mathrm{d}r \right\} P_m^{(-1/2,0)}(1-2\zeta^2)\mathrm{d}\zeta = \int_0^1 P_m^{(-1/2,0)}(1-2\zeta^2)\mathrm{d}\zeta = 1 \quad \text{for } m=0,$$

$$= 0 \quad \text{for } m \neq 0.$$

Appendix C

Expanding the product of Bessel functions in a series expansion up to second-degree terms and using Eq. (5.1), $A_{n,j}^{ms}$ in Eq. (4.9) are expressed as

$$
\begin{aligned}
A^{00}_{0,0} &= -2K + \mathrm{i}ak_2I_1 + \frac{4}{3\pi}a^2k_2^2I_2E + \mathrm{o}\left(k_2^3\right),\\
A^{00}_{0,1} &= A^{10}_{0,0} = -\frac{2}{15\pi}a^2k_2^2I_2E + \mathrm{o}\left(k_2^3\right),\\
A^{10}_{0,1} &= -\frac{2K}{5} - \frac{4}{21\pi}a^2k_2^2I_2E + \mathrm{o}\left(k_2^3\right).
\end{aligned}
\tag{C.1}
$$

Substituting $s = 0$, $m = 0$, 1, $n = 0$ and $j = 0$, 1 in Eq. (4.4), the following equations are obtained

$$
\begin{aligned}
A^{00}_{0,0}W^0_0 + A^{00}_{0,1}W^0_1 &= \frac{w_0\pi}{b},\\
A^{10}_{0,0}W^0_0 + A^{10}_{0,1}W^0_1 &= 0.
\end{aligned}
\tag{C.2}
$$

Solving the Eq. (C.2) using Eq. (C.1), the expression in Eq. (5.4) is obtained.

References

Awojobi, A.O., Grootenhuis, P., 1965. Vibration of rigid bodies on semi-infinite elastic media. Proc. R. Soc. London A 287, 27.

Chatterjee, M., Roy, A., 1990. On an elliptic punch indenting an elastic layer. Indian J. Pure Appl. Math. 21 (10), 935.

Gladwell, W.M.L., 1968. Forced tangential and rotatory vibration of rigid circular disc on a semi-infinite solid. Int. J. Engng. Sci. 6, 591.

Gradshteyn, I.S., Ryzhik, L.N., 1980. In: Jeffrey, A. (Ed.), Tables of Integrals, Series and Products. Academic Press, Orlando, FL.

Krenk, S., Schmidt, H., 1981. Vibrations of an elastic circular plate on an elastic half space – a direct approach. J. Appl. Mech. 48, 161.

Krenk, S., Schmidt, H., 1982. Asymmetric vibrations of a circular elastic plate on an elastic half space. Int. J. Solids Struct. 18 (2), 91.

Luco, J.E., Westmann, R.A., 1971. Dynamic response of circular footings. J. Engng. Mech. Div. ASCE97, 1381.

Mukherjee, M., 1998. Orthogonal polynomial solutions for a flat elliptic indenter on a multilayered elastic half space. Indian J. Pure Appl. Math. 29, 991–1010.

Robertson, I.A., 1965. Forced vertical vibration of a rigid circular disc on a semi-infinite elastic solid. Proc. R. Soc. London A 287, 27.

Roy, A., 1986. Dynamic response of elliptical footings. Int. J. Solids Struct. 22 (3), 293.

Stallybrass, M.P., Scherer, S.E., 1975. Forced vertical vibration of a rigid elliptical disc on an elastic half space. Int. J. Engng. Sci. 14, 511.